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Research Article

Measuring Discrepancies between Allocations: The Case of Loosemore-Hanby and Gini Indices

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Abstract

Inequality indices, such as the Gini index or the Schutz index, are applied in numerous scientific disciplines and their properties have been analyzed in multiple studies. On the other hand, disproportionality indices are known in relatively narrow disciplines and their properties were described much less frequently in comparison with inequality indices. This type of indices is worth considering more thoroughly for at least two reasons. Firstly, many inequality indices are special cases of disproportionality indices. Secondly, these indices can be a convenient tool to compare the structures of any pairs of allocations and, consequently, to measure their discrepancies. While measures of inequality enable the study of income inequalities or wealth inequalities, disproportionality indices can also be used to analyze inequalities between income and wealth distributions.

This paper proposes a disproportionality index whose particular case is the Gini index. Using the relationship between the Pietra/Schutz and Gini indices and the analogous relationship between the Loosemore–Hanby index and the constructed index, it is shown what properties it has and what kind of information this index can provide. These considerations are illustrated by an example of measuring the discrepancies between allocations taken into account as patterns within the planned reduction of pesticide sales in the European Union countries under the Farm to Fork Strategy.

Keywords: data science * disproportionality * Gini index * Pietra/Schutz/Hoover/Robin-Hood index.

JEL Classification: C43, D63

Introduction

The quantitative characteristics of analyzed phenomena can be obtained using some dedicated functions called indices, indicators or measures of nonfulfillment of the given studied phenomenon. These functions with nonnegative values are equal to zero if and only if a specific condition is fulfilled that is related to that index. Indices can be classified according to the area of their applications, their properties and their construction rules. The largest category of those functions includes inequality indices applied to

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compare income and wealth distributions in different populations.

'Equally' usually means the same amount allocated to everyone. Inequality indices are used to measure the extent of the difference between the given distribution and the equal distribution. What to do, however, if an equality pattern is different? The typical approach is to modify the input data and to replace it with relevant quotients. For example, equal paying for the work done can be comprehended as pay proportional to the number of hours worked. Then, instead of analyzing the inequalities of absolute pays, we may consider the pay inequalities converted to hours worked. Such an approach has some significant limitations. For example, it is impossible to consider the case when someone with zero hours worked receives a large pay or the case when someone with a positive number of hours worked does not receive any pay. Alternatively, in such cases, it may be recommended to use the disproportionality indices which fulfill specific conditions.

More than fifty different inequality indices are known (Coulter, 2018). The most popular inequality index is probably the Gini index, as suggested by at least several new papers every year which directly refer to this index in diverse research areas. There are two special reasons of this popularity. Firstly, the index can be interpreted graphically, thus allowing to intuitively interpret the obtained result as the ratio of the area that lies between the line of perfect equality and the Lorenz curve, over the total area under the line of equality. Secondly, over a dozen of variants of this index are known (see Yitzhak and Schechtman, 2013; Giorgi and Gigliarano, 2016), including in particular the variants requiring a relatively small number of elementary operations, such as the algorithm proposed by Sen (1973). The Schutz index, also known as the Pietra index, the Hoover index or the Robin Hood index, is less frequently used but it has many natural interpretations and many interesting applications.

Inequality indices are currently applied in multiple research areas, such as welfare economics, distributional analysis, information theory (Cowell, 2011) and many others (Sitthiyot and Holasut, 2020). Such types indices as indices of disproportionality, volatility, disparity, malapportionment, etc., are typically dedicated to particular research areas, e.g. the analysis of disproportionality of electoral systems. It is worth looking at this type of indices from a broader perspective, as a measure of the mismatch or inconsistency between any two allocations or two distributions. These allocations can be, for example, a pattern and its alternative, a portfolio status at two points in time, or two alternative descriptions of states of the studied phenomenon by means of finite vectors.

About twenty indices of disproportionality have been proposed so far (see Karpov, 2008; Chessa and Fragnelli, 2012), in particular the Rae index and the Loosemore-Hanby index. If a disproportionality index fulfills specific conditions and one of the allocations compared is the equal allocation, then the values of this disproportionality index are identical to the values of specified inequality index. In other words, disproportionality indices fulfilling specific conditions should be treated as generalized inequality indices. For example, the Loosemore-Hanby index can be considered a generalization of the Schutz index. This observation is the starting point for construction of the disproportionality index whose specific case is the Gini index. This construction is significantly different from foregoing attempts to use the Gini index as disproportionality index, e.g. the index proposed by Karpov (2008) who applies the quotients of respective coordinates of analyzed allocations which results in the lack of symmetry and the index's susceptibility to various paradoxes (Dniestrzański, 2015), (Dniestrzański and Łyko, 2020). The proposed construction does not show those undesirable properties.

The paper has the following structure. Section 3 defines the categories of inequality and disproportionality indices and describes the relationships between them. For this purpose, the definition of majorization relation currently accepted in the literature (see Section 2) should be modified. Section 4 describes the relationship between the Loosemore-Hanby index and the Schutz index, and the less-known relationship between the Schutz index and the Gini index, so that the Gini index can be seen as a peculiar modification of the Schutz index. The Loosemore-Hanby index is a percentage size of shares which should be transferred so that the structures of shares in the studied allocations were identical. Alternatively, if the group of agents is divided into two separate sub-groups based on the sizes of allocation shares, i.e. the group of agents whose differences of percentage shares are positive and the group of agents whose

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differences are negative, then the Loosemore-Hanby index measures the variation between those two groups and only between those two groups. The proposed construction considers additionally, similarly like the Gini index with the Schutz index, the variations in those two groups. This relationship is the underpinning of disproportionality index construction of which the Gini index is its special case. Chapter 5 describes the axioms of this indicator, while Chapter 7 indicates potential research problems related to its properties and its possible modifications.

Section 6 presents the case of discrepancy between two patterns considered as criteria of quota allocation in connection with planned reductions of pesticide sales under the Farm to Fork Strategy (EC, 2020). According to this strategy the amount of pesticide sales in respective European Union (EU) countries should be reduced until 2050 by 50 per cent, proportionately to the amount of pesticide sales in those countries in 2016. The farmers from many countries have protested against this strategy. One of the contested assumptions is that pesticide sales planning should take into account the state of the market at a given moment, instead of the area of agricultural land in individual countries. This case properly illustrates the application of the proposed index. Its value can be interpreted as the information about the scale of inequality in the development of agricultural technologies between the farmers in individual EU countries.

Framework and Notation

Let $n \in \mathbb{N}$ be a number, $N = \{1,2,\dots,n\}$ be set of individuals and vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ be allocation of goods or discrete distribution of income, where x_i denotes the allocation or income of the agent i, where $i \in N$. In other words, $\boldsymbol{x}$ is a n-element vector with nonnegative values summing up to a positive value. Let $\pi(\boldsymbol{x}) = (x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$ denote the change of the vector $\boldsymbol{x}$ ordering, i.e. the rearrangement of coordinates relative to the permutation π, or the function $\pi: N \to N$ which is injective and surjective (onto function).

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^n \backslash \{\boldsymbol{0}\}$ be two allocations. We say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are *comonotone* if $(x_i - x_j)(y_i - y_j) \geq 0$ for all $i, j \in N$. In other words, $\boldsymbol{x}$ and $\boldsymbol{y}$ are comonotone if there exists such an ordering set of individuals $\pi: N \to N$, that $x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}$ and $y_{\pi(1)} \leq y_{\pi(2)} \leq \cdots \leq x_{\pi(n)}$. For example, vectors $(15, 2, 0, 3)$ and $(4, 3, 1, 3)$ are comonotone because vectors (0,2,3,15) and (1,3,3,4) are comonotone.

For a given $\boldsymbol{x} \in \mathbb{R}_+^n \backslash \{\boldsymbol{0}\}$ let $\boldsymbol{x}^*$ denote *share vector*, which is a mean vector obtained from x by rearranging its elements in a non-decreasing way and normalized to: $x_1^* \leq x_2^* \leq \cdots \leq x_n^*$ and $x_1^* + x_2^* + \cdots + x_n^* = 1$. For example, if $\boldsymbol{x} = (2, 0, 5, 3)$, then $\boldsymbol{x}^* = \frac{1}{10}(0, 2, 3, 5) = (0, 0.2, 0.3, 0.5)$. The set of all share vectors is denoted by $\mathbb{S}^n$. Let $\boldsymbol{v}^k = (\underbrace{0,0,\dots,0}_{n-k}, \underbrace{\tfrac{1}{k},\dots,\tfrac{1}{k}}_{k})$, where $k \in N$. In particular, $\boldsymbol{v}^1 = (0,0,\dots,0,1)$, $\boldsymbol{v}^2 = (0,0,\dots,0,\frac{1}{2},\frac{1}{2})$, and $\boldsymbol{v}^n = (\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n})$.

Let $\mathrm{cumsum}(\boldsymbol{x})$, or in short $\mathrm{cs}(\boldsymbol{x})$, denote a vector of partial sums. If $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ then $\mathrm{cs}(\boldsymbol{x}) = (x_1, x_1 + x_2, \dots, x_1 + x_2 + \cdots + x_n)$. For example, $\mathrm{cs}(0, 0.2, 0.3, 0.5) = (0, 0.2, 0.5, 1)$.

Definition 1. For two comonotone vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^n \backslash \{\boldsymbol{0}\}$ we say that $\boldsymbol{x}$ *is majorized by* $\boldsymbol{y}$, denoted by $\boldsymbol{x} \prec \boldsymbol{y}$ or $\boldsymbol{y}$ *majorizes* $\boldsymbol{x}$, denoted by $\boldsymbol{y} \succ \boldsymbol{x}$, if $\sum_{i=1}^k x_i^* \leq \sum_{i=1}^k y_i^*$. Equivalently $\boldsymbol{x} \prec \boldsymbol{y}$ if $\mathrm{cs}(\boldsymbol{x}^* - \boldsymbol{y}^*) \leq \boldsymbol{0}$ and $\boldsymbol{y} \succ \boldsymbol{x}$ if $\mathrm{cs}(\boldsymbol{y}^* - \boldsymbol{x}^*) \geq \boldsymbol{0}$.

This definition is different from the standard definition of majorization (see Marshall, Olkin and Arnold, 2011; Arnold and Sarabia, 2018; Khan *et al.*, 2019). Firstly, comparison of shares instead of input vectors results in the fact that these vectors do not necessarily sum up to the same value. Secondly, the standard definition equates vectors with the same monotonic ordering, which is a preferred simplification for the inequality indices. For example, according to the classical definition, vectors (1, 0, 3) and (1,3,0) are identical with vector (0, 1, 3). In the case of disproportionality indices, this equation of vectors leads to the loss of information about comonotonicity for the analyzed allocations. The condition of comonotonicity of vectors is necessary so as to get the implication: if $\boldsymbol{x} \prec \boldsymbol{y}$ then $\boldsymbol{y} \succ \boldsymbol{x}$. For example, the allocations $\boldsymbol{x} = (1,4,5)$ and $\boldsymbol{y} = (2,5,3)$ satisfy $\boldsymbol{x} \prec \boldsymbol{y}$, but do not satisfy $\boldsymbol{y} \succ \boldsymbol{x}$. This happens because the ordering determined by the vector of the first allocation yields the difference of shares equal $\frac{1}{10}(\boldsymbol{x} - \boldsymbol{y}) = (0.1 - 0.2, 0.4 - 0.5, 0.5 - 0.3) = (-0.1, -0.1, 0.2)$ and $\mathrm{cs}(\boldsymbol{x}^* - \boldsymbol{y}^*) = (-0.1, -0.2, 0) \leq (0, 0, 0)$. However, the ordering determined by the vectors of the second allocation yields $\frac{1}{10}(\boldsymbol{y} - \boldsymbol{x}) = (0.2 - 0.1, 0.3 - 0.5, 0.5 - 0.4) = (0.1, -0.2, 0.1)$, and thus $\mathrm{cs}(\boldsymbol{y}^* - \boldsymbol{x}^*) = (0.1, -0.1, 0) \ngeq (0, 0, 0)$.

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Relation between Disproportionality and Inequality Indices

In political science, the disproportionality index is a tool that allows us to numerically determine the extent to which the effect of a given electoral procedure differs from the allocation proportional to the population of eligible voters. Generally, the index of disproportionality is a function numerically expressing how much the two vectors are not proportional. Similarly to other types of indexes, different authors provide different sets of axioms or properties that such an index should satisfy. For example, eight such criteria are discussed in Koppel and Diskin (2009) and five in Karpov (2008). Each selection of conditions is basically subjective and depends on the field of application. This study postulates the following set of conditions which should be fulfilled by disproportionality indices.

Definition 2. Disproportionality index is a continuous function $f: \mathbb{R}_+^n \backslash \{\mathbf{0}\} \times \mathbb{R}_+^n \backslash \{\mathbf{0}\} \to \mathbb{R}$ subject to the following conditions:

- Non-negativity: $f(\boldsymbol{x}, \boldsymbol{y}) \geq 0$,
- Proportionality: $f(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \lambda \boldsymbol{y}$,
- Symmetry: $f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x})$,
- Scale Independent: $f(\lambda \boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{x}, \boldsymbol{y})$,
- Anonymity: $f\big(\pi(\boldsymbol{x}), \pi(\boldsymbol{y})\big) = f(\boldsymbol{x}, \boldsymbol{y})$ for all permutation $\pi: N \to N$,
- Schur Convex: if $\boldsymbol{x} \prec \boldsymbol{y} \prec \boldsymbol{z}$ then $f(\boldsymbol{x}, \boldsymbol{y}) \leq f(\boldsymbol{x}, \boldsymbol{z})$ and $f(\boldsymbol{x}, \boldsymbol{z}) \leq f(\boldsymbol{y}, \boldsymbol{z})$

for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}_+^n \backslash \{\mathbf{0}\}$ and $\lambda > 0$.

Due to the conditions of non-negativity, symmetry and Schur convexity, the disproportionality indices can be treated as the equivalents of distance. Scale independence and anonymity are typical conditions of most indices. Scale independence means that the value of function does not depend on the choice of metric unit or the total sum of individual components of analyzed vectors. This condition is usually met by properly scaling the function or, equivalently, by scaling the vector to obtain the share vector. Anonymity condition means that the value of function does not depend on the order of data input and that relevant data are correctly compared, i.e. the first component of the first vector is compared with the first component of the second vector, the second component of the first vector is compared with the second component of the second vector, and so on. If analyzed data represent a matrix of two rows, then the condition of symmetry indicates that rearranging the rows of this matrix does not change the value of the index and, similarly, the anonymity condition implies that rearranging the columns does not matter.

The Schur-convexity condition is equivalent to the Pigou-Dalton principle of transfers or a finite sequence of "Robin-Hood operations". In other words, disproportionality indices are a subclass of Schur-convex functions with respect to each variable separately. Majorization can be generalized to the Lorenz ordering. In a set of non-negative vectors with a fixed number of elements and no zero vector, the relation of majorization is reflexive and transitive. The conditions of homogeneity and anonymity imply that any non-negative vector $\boldsymbol{x}$ can be equated with a share vector $\boldsymbol{x}^*$. In a set of share vectors $\mathbb{S}^n$ the relation of majorization is an order and for all $\boldsymbol{x}^* \in \mathbb{S}^n$ $\boldsymbol{v}^n \prec \boldsymbol{x}^* \prec \boldsymbol{v}^1$, i.e. a poset $(\mathbb{S}^n, \prec)$ is a lattice. In particular, any $\boldsymbol{x} \in \mathbb{R}_+^n \backslash \{\mathbf{0}\}$ is comonotone with respect to $\boldsymbol{v}^n$. As a result, a disproportionality index defined in this way generates an inequality index and also an equality index. It suffices to assume an extremely egalitarian vector (i.e. a vector whose share vector is $\boldsymbol{v}^n$) or an extremely elitist vector (i.e. a vector whose share vector is $\boldsymbol{v}^1$) as one of the two arguments.

Fact 1. If a function $f: \mathbb{R}_+^n \backslash \{\mathbf{0}\} \times \mathbb{R}_+^n \backslash \{\mathbf{0}\} \to \mathbb{R}$ is a disproportionality index, then functions $g_n: \mathbb{R}_+^n \backslash \{\mathbf{0}\} \to \mathbb{R}$ and $g_1: \mathbb{R}_+^n \backslash \{\mathbf{0}\} \to \mathbb{R}$ given by the formula $g_n(\boldsymbol{x}) = f(\boldsymbol{x}, \boldsymbol{v}^n)$ and $g_1(\boldsymbol{x}) = f(\boldsymbol{x}, \boldsymbol{v}^1)$ fulfill the following conditions:

- Non-negativity: $g_n(\boldsymbol{x}) \geq 0$ and $g_1(\boldsymbol{x}) \geq 0$,
- Extremely-egalitarian zero: $g_n(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x}^* = \boldsymbol{v}^n$ and extremely-elitism zero: $g_1(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x}^* = \boldsymbol{v}^1$,
- Scale Independent: $g_n(\lambda \boldsymbol{x}) = g_n(\boldsymbol{x})$ and $g_1(\lambda \boldsymbol{x}) = g_1(\boldsymbol{x})$,
- Anonymity: $g_n(\pi(\boldsymbol{x})) = g_n(\boldsymbol{x})$ and $g_1(\pi(\boldsymbol{x})) = g_1(\boldsymbol{x})$ for all permutation $\pi: N \to N$,
- Schur Convex: $\boldsymbol{x} \prec \boldsymbol{y} \Rightarrow g_n(\boldsymbol{x}) \leq g_n(\boldsymbol{y})$ and $g_1(\boldsymbol{x}) \geq g_1(\boldsymbol{y})$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^n \backslash \{\mathbf{0}\}$ and $\lambda > 0$.

Functions $g_n(\boldsymbol{x})$ and $g_1(\boldsymbol{x})$ can be termed an inequality index and equality index, respectively.

The above construction directly yields: $g_n(\boldsymbol{v}^1) = g_1(\boldsymbol{v}^n)$ and if $\boldsymbol{x}^* \neq \boldsymbol{v}^1$ and $\boldsymbol{x}^* \neq \boldsymbol{v}^n$ then $0 < g_n(\boldsymbol{x}^*) < g_n(\boldsymbol{v}^1)$ and $0 < g_1(\boldsymbol{x}^*) < g_1(\boldsymbol{v}^n)$.

The indices $g_n(\boldsymbol{x})$ and $g_1(\boldsymbol{x})$ generated by the same disproportionality index will be called the associated indices. The examples of such indices are presented in Table 1. The value $g_n(\boldsymbol{v}^1) = g_1(\boldsymbol{v}^n)$ is a maximum value or the span of these indices. This parameter is useful for scaling the indices. The highest value of disproportionality indices is reached when the opposite vectors are compared, i.e. the vectors $(0,0,\dots,0,1)$ and $(1,0,0,\dots,0)$. For example, the span of equality and inequality indices based on the least squares index tends to $1/\sqrt{2}$ when the number of analyzed data tends to infinity. Therefore, it is more favorable to multiply this index by $\sqrt{2}$, so that the maximum possible value of associated indices could be limited by 1.

Table 1. Examples of inequality indices and equality indices – special cases of disproportionality indices

Name	$f(\boldsymbol{x},\boldsymbol{y})$	$g_n(\boldsymbol{x}) = f(\boldsymbol{x},\boldsymbol{v}^n)$	$g_1(\boldsymbol{x}) = f(\boldsymbol{x},\boldsymbol{v}^1)$	$g_1(\boldsymbol{v}^n)$
Maximum deviation	$\max_{i\in N}\lvert x_i^* - y_i^*\rvert$	$x_n^* - \frac{1}{n}$	$\max\{x_{n-1}^*, x_n^*\}$	$\frac{n-1}{n}$
Loosemore-Hanby	$\frac{1}{2}\sum_{i\in N}\lvert x_i^* - y_i^*\rvert$	$\frac{1}{2}\sum_{i\in N}\left\lvert x_i^* - \frac{1}{n}\right\rvert$	$1 - x_n^*$	$\frac{n-1}{n}$
Least square or Gallagher	$\sqrt{\frac{1}{2}\sum_{i\in N}(x_i^* - y_i^*)^2}$	$\frac{1}{\sqrt{2}}\sqrt{\sum_{i\in N} x_i^{*2} - \frac{1}{n}}$	$\frac{1}{\sqrt{2}}\sqrt{1 + \sum_{i\in N} x_i^{*2} - 2x_n^*}$	$\sqrt{\frac{n-1}{2n}}$

Source: Author's own study

Relationships between the Schutz and Gini Indices

A special case of the Loosemore-Handby index is the Schutz index, also called the Pietra index, Hoover index or Robin Hood index. When this index is properly bounded, it becomes equal or complementary to three other indices: the relative mean deviation, Wilcox's average deviation analog, and Martin and Gray's relative variation (Coulter, 2018). Different names of this index used in the literature result from the fact that various researchers were introducing it in different times past, often to examine diverse issues. (Pietra in 1915, Hoover in 1936, Schutz in 1951 and Atkinson, Micklewright in 1991). There are several interpretations of this index. First, it is a measure of total surplus over the mean. The value of the index is calculated by subtracting the mean values, i.e. the vector $\boldsymbol{v}^n$, from the share vector of the analyzed vector, and adding up the positive values obtained (i.e. the sum of surpluses over the average) (see Fig. 1). Alternatively, instead of adding up the surpluses above the mean value, we can calculate the sum of deficits below this value, or half of the sum of absolute differences

$$\mathrm{LHi}(\boldsymbol{x},\boldsymbol{v}^v) = \mathrm{Schutz}(\boldsymbol{x}) = \sum_{x_i^* > \frac{1}{n}}\left(x_i^* - \frac{1}{n}\right) = \sum_{x_i^* < \frac{1}{n}}\left(\frac{1}{n} - x_i^*\right) = \frac{1}{2}\sum_{i\in N}\left\lvert x_i^* - \frac{1}{n}\right\rvert.$$

Figuratively, the value of this index represents the percentage of total goods that should be taken from the rich, i.e. those who have more than the average, and transfer to them the poor so as to reach the perfect equality status.

An alternative interpretation of this index is based on the Lorenz curve of concentration. A vector of partial sums of an n-element share vector represents the cumulative discrete distribution function of variable defined at n points. If these points are $\frac{1}{n}, \frac{2}{n}, \dots, 1$ (i.e. the points representing the values of the vector of partial sums $\boldsymbol{v}^n$), and the analyzed vector was monotonically ordered, then a polygonal chain connecting the values of the vector of partial sums determines the Lorenz curve. Any Lorenz curve is limited by the Lorenz curve of vector $\boldsymbol{v}^n$, as a consequence of the fact that vector $\boldsymbol{v}^n$ is

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majorized by any share vector $\boldsymbol{x}^*$. Let us determine the vector of cumulative differences of shares between the given vector and an equal vector $\mathrm{cs}(\boldsymbol{v}^n - \boldsymbol{x}^*)$. Respective coordinates of this vector correspond to the lengths of segments between the Lorenz curve of vector $\boldsymbol{x}$ and the Lorenz curve of the equal vector (see Fig. 2). These lengths obviously increase originally and then they decrease to 0. The Schutz index returns the maximum value of cumulative differences:

$$\mathrm{Schutz}(\boldsymbol{x}) = \frac{1}{2}\sum_{i\in N}\left|\frac{1}{n} - x_i^*\right| = \max \mathrm{cs}(\boldsymbol{v}^n - \boldsymbol{x}^*).$$

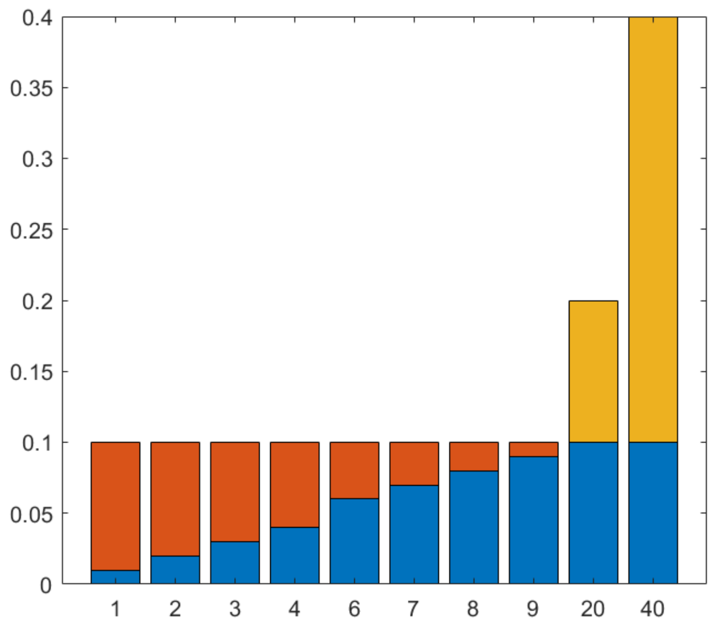

Fig. 1 Graphical illustration of the Schutz index as a total surplus, or a total deficit, of shares with respect to the mean values

The strength of the Schutz index is its simplicity. Its weakness is that it does not take good advantage of all available information. As it represents the percentage surplus above the mean value, any possible inequality below or above this value is not taken into account. Any change of the value of vector below its mean that does not result in the change of the mean itself (or similar change of the value above the mean) does not affect the value of the index.

The Gini index can be used to evaluate such additional variation that was not reflected by the Schutz index. This index can be identified with the doubled area limited by the Lorenz concentration curve and the line of ideal equality. This figure can be divided into n figures with the same width, whereas the first figure and the last figure are the triangles with altitudes equal respectively to the first and next-to-last coordinate of the vector of cumulative differences of shares between the equal vector and the analyzed vector. Other figures are trapezoids whose parallel sides have lengths equal to respective pairs of coordinates of this vector (see Fig. 2). These coordinates are denoted by $cs_1, cs_2, \dots, cs_{n-1}, cs_n$, where $cs_n = 0$. Adding up the areas of these figures yields:

$$\frac{1}{n}\frac{cs_1}{2} + \frac{1}{n}\frac{cs_1 + cs_2}{2} + \cdots + \frac{1}{n}\frac{cs_{n-2} + cs_{n-1}}{2} + \frac{1}{n}\frac{cs_{n-1}}{2} = \frac{1}{n}(cs_1 + cs_2 + \cdots + cs_n).$$

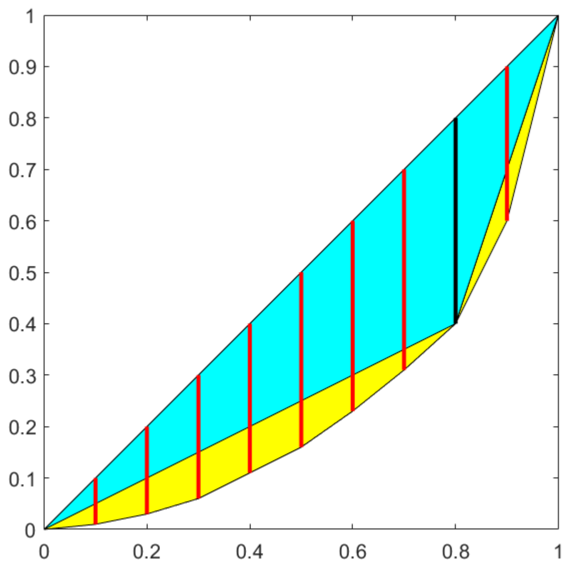

Fig. 2 Graphical illustration of the Schutz index and the Gini index as the maximum distance and the doubled area between the Lorenz curve and the line of ideal equality

In other words, to calculate the value of the Gini index for the vector $\boldsymbol{x}$, we have to determine the vector of difference of shares of two vectors; the equal vector and the vector $\boldsymbol{x}$. Next, we calculate the vector of cumulative sums of these differences. The Gini index is a doubled arithmetic mean of the vector of cumulative differences.

$$\mathrm{Gini}(\boldsymbol{x}) = \frac{2}{n}\sum_{k=1}^{n}\sum_{i=1}^{k}(v_i^n - x_i^*) = \frac{2}{n}\sum_{k=1}^{n}\left(\frac{k}{n} - \sum_{i=1}^{k} x_i^*\right).$$

Error! Reference source not found. Fig. 2 illustrates an important relation between the Schutz and Gini indices. Even though it is easy to deduce, it is not widely known, as evidenced by the fact that it is not mentioned in the papers analyzing the relationships between these indices (see Allen, 2022). Both indices are functions whose arguments are the coordinates of the vector of accumulation of share differences. Consider a set of vectors whose value of the Schutz index is the same and equals h, where $0 < h \leq \frac{n-1}{n}$. Different Lorenz curves represent these vectors, and thus, the different values of the Gini index. The vectors whose Lorenz curves are polygonal chains connecting the points (0,0), $(\frac{k}{n}, \frac{k}{n} - h)$ and (1,1), where $k < n$ (see Fig. 2), have the minimum value of this index. This value does not depend on k and equals h, i.e. it equals the value of the Schutz index. The vectors representing this curve have two different values only and their shares equal $\frac{1}{n} - \frac{h}{k}$ and $\frac{1}{n} + \frac{h}{n-k}$, respectively. Each vector with a greater number of values (or a vector whose shares have other values) is represented by the Lorenz curve with greater area, thus, its Gini index will have the value higher than h. This leads to the following fact 2.

Fact 2. When dividing the values of a non-negative vector $\boldsymbol{x}$, which is not equal to a zero vector, in two disjoint groups, one with the values below the mean value, and the other with values above or equal to the mean value, then Schutz($\boldsymbol{x}$) indicates the variation between these groups, whereas Gini($\boldsymbol{x}$) additionally represents the variations within these groups.

The Gini Index Analog in a Family of Disproportionality Indices and its Properties

The Schutz index and the Gini index can be considered numerical expressions of how much a given vector differs from the equal vector. The method to obtain these indices explained above can be generalized by replacing the equal vector with any vector of non-negative values that is not equal to a zero vector. The sequence of determining a vector of partial sums of share differences for these vectors is a significant issue. In the case of the Schutz and Gini indices, this sequence conforms the monotone arrangement of the vector's elements; whereas, when construing disproportionality indices, the analyzed vectors do not have to be comonotone and adding up the elements in such sequence could yield undesirable effects.

In order to avoid it, we should determine a vector of share differences of analyzed vectors before calculating the vector of partial sums, then the obtained vector can be sorted monotonically from the highest to the lowest value. The maximum of the vector of partial sums sorted nonincreasingly is the value of the Loosemore-Hanby index and the doubled arithmetic mean of the partial sums is an analog of the Gini index.

Algorithm

Input:
$\boldsymbol{x} = (x_1, x_2, \ldots, x_n), \boldsymbol{y} = (y_1, y_2, \ldots, y_n)$, where $x_i, y_i \geq 0$ for all $i \in N$ and $\sum_{i=1}^{n} x_i > 0$, $\sum_{i=1}^{n} y_i > 0$.

Output:
$\mathrm{LHi} = \mathrm{LHi}(\boldsymbol{x}, \boldsymbol{y})$ – the Loosemore-Handby index,
$\mathrm{Gi} = \mathrm{Gi}(\boldsymbol{x}, \boldsymbol{y})$ –the Gini index analog.
Step 1. $\boldsymbol{x}' = \boldsymbol{x}/\mathrm{sum}(\boldsymbol{x})$; $\boldsymbol{y}' = \boldsymbol{y}/\mathrm{sum}(\boldsymbol{y})$
Step 2. $\boldsymbol{z} = \mathrm{sort}(\boldsymbol{x}' - \boldsymbol{y}', \text{"descent"})$
Step 3. $\boldsymbol{cs} = \mathrm{cumsum}(\boldsymbol{z})$
Step 4. $\mathrm{LHi} = \max(\boldsymbol{cs})$, $\mathrm{Gi} = 2 * \mathrm{mean}(\boldsymbol{cs})$.

A simple consequence of the fact that the vector $\boldsymbol{v}^n$ majorizes every non-negative non-zero vector is that the coordinates of the vector $\boldsymbol{v}^n - \boldsymbol{x}^*$ form a non-increasing sequence. Hence, sorting the coordinates in non-increasing order is a neutral operation. This means that the Schutz index is a special case of the Loosemore-Hanby index and the Gini index is a special case of the proposed index.

$\mathrm{LHi}(\boldsymbol{x}, \boldsymbol{v}^n) = \mathrm{Schutz}(\boldsymbol{x})$ and $\mathrm{Gi}(\boldsymbol{x}, \boldsymbol{v}^n) = \mathrm{Gini}(\boldsymbol{x})$.

These equalities can be formulated more generally:

if $\boldsymbol{x} \prec \boldsymbol{y}$ then $\mathrm{LHi}(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{Schutz}(\boldsymbol{y}) - \mathrm{Schutz}(\boldsymbol{x})$ and $\mathrm{Gi}(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{Gini}(\boldsymbol{y}) - \mathrm{Gini}(\boldsymbol{x})$.

Similarly to the Loosemore-Hanby index, the proposed index based on the Gini index has all the properties of a disproportionality index formulated in Definition 2. Both indices satisfy also the equivalent axiom of population independence, which is required of inequality indices, i.e.

$\mathrm{LHi}(\boldsymbol{x} \sqcup \boldsymbol{x}, \boldsymbol{y} \sqcup \boldsymbol{y}) = \mathrm{LHi}(\boldsymbol{x}, \boldsymbol{y})$ and $\mathrm{Gi}(\boldsymbol{x} \sqcup \boldsymbol{x}, \boldsymbol{y} \sqcup \boldsymbol{y}) = \mathrm{Gi}(\boldsymbol{x}, \boldsymbol{y})$,

where $\boldsymbol{x} \sqcup \boldsymbol{x}$ is the union of $\boldsymbol{x}$ with a copy of itself. The Loosemore-Hanby index also satisfies the property that its values are invariant under inclusion of zero values. An index based on the Gini index does not satisfy this property because the arithmetic mean depends on the number of coordinates.

$\mathrm{LHi}(\boldsymbol{x} \sqcup \boldsymbol{0}, \boldsymbol{y} \sqcup \boldsymbol{0}) = \mathrm{LHi}(\boldsymbol{x}, \boldsymbol{y})$ and $\mathrm{Gi}(\boldsymbol{x} \sqcup \boldsymbol{0}, \boldsymbol{y} \sqcup \boldsymbol{0}) > \mathrm{Gi}(\boldsymbol{x}, \boldsymbol{y})$,

where $\boldsymbol{0}$ is either the number zero or a zero vector.

Analogously to the relationship between the Schutz index and the Gini index (see *Fact 2*), the values of the proposed index are always higher than or at most equal to the Loosemore-Hanby index. The coordinates of vectors represent a part of allocation per individual agents, and, if the group of agents can be divided in two separate subgroups with respect to the sizes of the allocation shares, i.e. the group of agents whose differences of share percentages are favorable and the group of other agents, then the Loosemore-Hanby index measures the variation between those two groups and only between those two groups. The proposed index considers additionally, just as the Gini index, the variations within those two groups.

Application

This section presents the results of application of the proposed index against the background of the Loosemore-Hanby index, using the example of selecting the benchmark in relation to the planned reductions in the volume of pesticides sold under the Farm to Fork Strategy (EC, 2020). According to this strategy, the amount of pesticide sales in respective European Union (EU) countries was to be reduced until 2050 by 50 per cent, proportionately to the amount of pesticide sales in those countries in 2016. The strategy led to farmers' protests in a number of EU countries. Farmers' income depends mainly on the level of specialization and application of appropriate technologies. More specialization and more advanced technologies applied in the cycle of agricultural production lead to more pesticide use. Farmers in the countries with much lower pesticide use than in countries where agricultural production is highly specialized are afraid that the introduction of quantitative restrictions of pesticide availability will result in the perpetuation of existing inequalities in income from running a business in agriculture. These protests were so intense that they indirectly led to the announcement of the withdrawal of this initiative by the European Commission before the EP elections in 2024 (EC, 2024).

Table 2. Differences between the structure of shares in pesticide sale and the structure of farmland in the EU countries in 2016 according to EUROSTAT

country	pesticide share	area share	difference	cumsum
Italy	0.1615	0.0686	0.0928	0.0928
Spain	0.2072	0.1434	0.0638	0.1567
Germany	0.1263	0.1019	0.0244	0.1810
France	0.1941	0.1739	0.0203	0.2013
Belgium	0.0177	0.0033	0.0144	0.2157
Netherlands	0.0290	0.0155	0.0135	0.2292
Portugal	0.0263	0.0169	0.0094	0.2386
Cyprus	0.0029	0.0005	0.0024	0.2409
Finland	0.0124	0.0110	0.0014	0.2423
Slovenia	0.0031	0.0024	0.0007	0.2430
Malta	0.0003	0.0001	0.0002	**0.2432**
Luxembourg	0.0005	0.0010	-0.0004	0.2428
Croatia	0.0050	0.0074	-0.0024	0.2404
Greece	0.0127	0.0163	-0.0036	0.2368
Denmark	0.0070	0.0134	-0.0065	0.2304
Estonia	0.0019	0.0095	-0.0076	0.2228
Austria	0.0117	0.0203	-0.0086	0.2142
Latvia	0.0046	0.0133	-0.0087	0.2055
Hungary	0.0263	0.0354	-0.0091	0.1963
Czechia	0.0160	0.0257	-0.0097	0.1866
Sweden	0.0053	0.0175	-0.0122	0.1744
Slovakia	0.0056	0.0190	-0.0134	0.1611
Poland	0.0658	0.0798	-0.0139	0.1471
Lithuania	0.0091	0.0240	-0.0149	0.1322
Ireland	0.0084	0.0309	-0.0226	0.1097
Bulgaria	0.0101	0.0571	-0.0469	0.0627
Romania	0.0291	0.0919	-0.0627	0.0000

Source: Author's own study

Is the level of inequality between the structure of pesticide sales and the structure of farmland a negligible question, used politically only to arouse emotions in certain groups of potential voters? Or is the level of this inequality high enough to cause long-term social tensions? The answers to these questions should be known before undertaking the activities which can lead to such potential strains. Different indices will return different results and their interpretation depends on the experience of decision makers. One has also to consider the fact that there are some significant differences between statistics collected by various international organizations, resulting from diverse definitions of pesticide and area of farmland adopted by these organizations. Let us accept the EUROSTAT terms and data (Eurostat, 2025), for the sake of simplicity. Table 2 presents the distributions of pesticide market and farmland area in the EU countries. Disproportionality between these distributions measured by the Loosemore-Hanby index equals 24.32%. This implies that after separating two disjoint categories of the EU countries with respect to the pesticide use, i.e. those countries whose percentage share in the pesticide use is greater that their percentage share in the farmland area, and those remaining, then in order to close the gap between the two categories, one has to allocate almost a quarter of total pesticides. The Loosemore-Hanby index applied does not regard current inequalities in both disjoint categories of countries. For example, farmers from Poland compare their own pesticide use per unit of farmland area with that of farmers from the Netherlands or Italy, where pesticide use is proportionally much greater, while farmers from Romania compare their situation not only with that in the Netherlands or Italy, but also with that in Austria or Poland. And therefore, applying the proposed index, i.e. the generalized Gini index, is more appropriate now. Its value is 37.39%, i.e. the inequalities in the structure of pesticide use in regard to the structure of farmland area exceed one third. Taking into account the experience from the period 2022–2024, the inequalities between the considered patterns should be considered significant.

Discussion

This section introduces some forthcoming research issues connected with the analysis of properties of the proposed index. The Gini index is a function whose argument is a vector with a fixed number of non-negative coordinates. Since linearly dependent vectors or the equivalent vectors with permutation accuracy have the same value of this index, the analysis of its properties can be reduced to the analysis on the set of share vectors. The proposed index can be analyzed as the Gini index with the extended domain. The arguments of this function are two vectors, which do not have to be comonotone, hence analogous restriction to the share vectors is not possible. The first research problem is connected with the set of values of this index and with its potential scaling. When the analyzed vectors are opposite or almost opposite, the value of the proposed vector can be higher than 1, and in extreme cases, as with vectors $(0,0,\dots,0,1)$ and $(1,0,\dots,0,0)$, it can tend to 2.

$$0 \le \mathrm{Schutz}(\boldsymbol{x}) \le \tfrac{n-1}{n}, \qquad 0 \le \mathrm{Gini}(\boldsymbol{x}) \le \tfrac{n-1}{n},$$
$$0 \le \mathrm{LHi}(\boldsymbol{x},\boldsymbol{y}) \le 1,\ 0 \le \mathrm{Gi}(\boldsymbol{x},\boldsymbol{y}) \le 2\tfrac{n-1}{n}.$$

The inequality indices with the value exceeding 1 are generally undesirable because their interpretation is problematic. In the case of the discussed indicator, exceeding the value of 1 is information about the opposition or extreme divergence of the analyzed vectors. Usually, its values are much lower than 1. Moreover, if one of the vectors is majorized by the other, the value of the discussed disproportionality indices does not exceed the value of the inequality indices. If $\boldsymbol{x} \prec \boldsymbol{y}$ then $\mathrm{LHi}(\boldsymbol{x},\boldsymbol{y}) \le \mathrm{Schutz}(\boldsymbol{x}) \le \mathrm{Schutz}(\boldsymbol{y})$ and $\mathrm{Gi}(\boldsymbol{x},\boldsymbol{y}) \le \mathrm{Gini}(\boldsymbol{x}) \le \mathrm{Gini}(\boldsymbol{y})$. Because of $\boldsymbol{v}^n \prec \boldsymbol{x} \prec \boldsymbol{y}$, the vector $\mathrm{cs}(\boldsymbol{y}^* - \boldsymbol{v}^n)$ is not lexicographically smaller than vector $\mathrm{cs}(\boldsymbol{x}^* - \boldsymbol{v}^n)$, which is not lexicographically smaller than vector $\mathrm{cs}(\boldsymbol{y}^* - \boldsymbol{x}^*)$; all the more, these inequalities are preserved by respective maximum values of these vectors as well as by their doubled arithmetic means. As a consequence, any scaling seems unnecessary, although this problem certainly needs more research.

The second issue is about scaling of the Loosemore-Hanby index and of the proposed index in the case when distributions with many negative values are permitted. In case of inequality indices, such as the Gini or Schutz indices, appropriate scaling methods are then used so that they do not exceed the value of 1 (see Raffinetti and Siletti, 2015; Ostasiewicz and Vernizzi, 2017; Brakel and Lok, 2021; Lee and Suh, 2025). Can these methods be adapted to the proposed index? Intuition suggests that yes, because these indices differ from the corresponding inequality indices in a potentially different ordering of the considered differences in share structures, but a formal proof has not yet been carried out.

The third potential problem regards the decomposition of disproportionality indices, which satisfy the conditions of Definition 2. The comparison of the proposed index with the Loosemore-Hanby index provides information about the decomposition into two specific sets, i.e. the set of agents for which the share difference set is positive, and separately for those for which such a set is negative. The form of decomposition into other types of subsets remains an open question, either by means of extending or modifying standard decomposition methods of the Gini index such as the methods described by Fleurbaey *et al.* (2024).

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