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Investigation of Alternative Serendipity Models for Solving the Problem of Torsion of Prismatic Rods*

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Abstract

The article discusses the testing of new alternative models of the biquadratic finite element of the serendipity family using the problem of torsion of a non-circular cross-section rod and compares the results obtained with the exact solution. The conversion of the Lagrange model to the serendipity model is undoubtedly a useful procedure that has been known for over fifty years. However, not all results of such a transformation satisfy users, especially supporters of physical interpretations. This concerns the value of nodal loads of uniform force (mass) (load 'spectrum'). For a long time, it was believed that there was a single basis for each serendipity finite element – a standard one, which was obtained algebraically. Using a new approach that employs a combined algebraic-geometric method for constructing basis functions on serendipity finite elements, it has been possible for the first time to obtain alternative bases with a control parameter on a biquadratic finite element. The presence of a parameter in the basis functions of serendipity finite elements allows optimising the computational qualities of the obtained alternative models. The results show that in the problem of torsion of a square section rod using the finite element method, when using new alternative models of the biquadratic finite element, we can obtain higher accuracy compared to the known standard model of this element. Alternative bases of the biquadratic serendipity finite element also have an advantage over the traditional triangulation procedure, since more triangular finite elements must be used to obtain the specified accuracy.

Keywords: Approximation, Serendipity finite elements, Finite element method, Alternative models, Dirichlet problem for the Poisson equation, Ritz method.

Introduction

The problem of torsion of a rod of arbitrary cross section is a classical problem of elasticity theory (Timoshenko, 1972). This problem was solved by S. Coulomb, T. Jung, L. Navier, and O. Cauchy (Timoshenko, 1957). The basic equations of the torsion problem of a prismatic rod of arbitrary cross section were obtained by the methods of elasticity theory by B. Saint-Venant and presented in 1853 to the Paris Academy of Sciences (Timoshenko, 1972; Timoshenko, 1957; Saint-Venant, 1961). An experimental method for solving the problem of rod torsion

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on the basis of the membrane analogy was proposed by L. Prandtl in 1903. (Timoshenko, 1972; Timoshenko, 1957). The calculation of the limiting torque using the sand embankment analogy was obtained by A. Nadai, E. Trefftz in 1923. The solutions of a large number of torsion problems for homogeneous and inhomogeneous bodies of constant and variable cross sections are given in the monograph by N.H. Harutyunyan and B.L. Abramyan (Harutyunyan and Abramyan, 1963). The general solution of the problem of torsion of a rod by means of the theory of functions of a complex variable is presented in the monograph by N.I. Muskhelishvili, the first edition of which was published in 1933. The theory of functions of a complex variable was first applied to the solution of the plane problem of the theory of elasticity by G.V. Kolosov in 1908-1909 (Timoshenko, 1957). When solving the problem of torsion of rods, numerical methods of calculation are effectively used: the finite difference method, the method of best products, and the method of variational iterations (Ilyin, Karpov and Maslennikov, 1990).

The solution of the problem by finite element methods in the form of the Ritz method, the Bubnov-Galerkin method, and the Vlasov-Kantorovich method is of interest (Ilyin, Karpov and Maslennikov, 1990). In (Segerlind, 1979), the problem of torsion of a rod of noncircular cross section is considered and a detailed solution by the finite element method with partitioning into triangular elements is given. The same problem can be solved using quadrilateral finite elements of the Serendipity family.

Literature Review

The Serendipity finite elements (SFE) originated more than 40 years ago in connection with the application of the point transformation of a quadrilateral to a standard square followed by the use of isoparametric approximation. A two-dimensional serendipity FE is a quadrilateral with nodes on the boundary and a system of basis functions that corresponds to these nodes. For a long time, it was believed that there was a single basis function on each serendipity finite element, the standard basis, which was obtained in an algebraic way (Segerlind, 1979). In (Astionenko, Litvinenko and Khomchenko, 2009), the authors show a new combined algebraic-geometric method for constructing basis functions on serendipitous finite elements. Using this method, they succeeded for the first time in obtaining alternative basis functions with a control parameter on a biquadratic serendipitous finite element (Q8). The presence of the parameter in the SFE Q8 functions allows us to optimize the computational qualities of the obtained alternative models. Of particular interest are non-standard serendipitous elements providing physically adequate integral characteristics

Purpose of The Study

Testing alternative serendipitous models of the quadratic FE using the problem of torsion of a non-circular cross-section rod and comparing the results obtained with the exact solution (Timoshenko, 1972).

Research Material

Torsion is a deformation of a rod characterized by mutual rotation of the cross sections relative to each other around the axis of the rod under the action of a pair of forces applied to its ends. In this case, only one internal force factor occurs in the cross-sections of the body – torque M. The torque is proportional to the volume encompassed by the stress surface φ. As a result of the action of torques, two cross-sections of a rod, which are at a certain distance, are rotated by an angle θ, which is called the angle of twist. Shafts and tension-compression springs work in torsion (Timoshenko, 1972).

Let us consider the problem of torsion of a rod of square cross-section, which is given in (Segerlind, 1979). Let be given: shear modulus $\mu = 0{,}8 \cdot 10^7\ N/cm^2$; relative twist angle (torsion angle) $1\ degree\ per\ 100\ cm$; cross-sectional area $1\,cm^2$; $K_{xx} = K_{yy} = 1$. Find the stress surface and torque arising from free torsion of the rod.
In mathematical physics, this problem is known as the Dirichlet problem for the Poisson equation (Ilyin, Karpov and Maslennikov, 1990):

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -2\mu\theta, \ . \ \varphi\big|_{\Gamma} = 0 \qquad (1)$$

The variational formulation of the rod torsion problem is related to the minimization of the functional:

$$\chi = \int_V \left(\frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \varphi}{\partial y}\right)^2 - (2\mu\theta)\varphi \right) dV \, . \tag{2}$$

The functional can be written in the form:

$$\chi = \int_V \left(\frac{1}{2}\{g\}^T [D]\{g\} - (2\mu\theta)\varphi \right) dV \, , \tag{3}$$

where $\{g\} = \begin{Bmatrix} \dfrac{\partial \varphi}{\partial x} \\ \dfrac{\partial \varphi}{\partial y} \end{Bmatrix}, [D] = \begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix}$.

Minimizing χ by $\{\Phi\}$ leads to a system of linear equations:

$$\sum_{e=1}^{E} \int_{V^{(e)}} \left[g^{(e)}\right]^T \left[D^{(e)}\right]\left[g^{(e)}\right] dV = \sum_{e=1}^{E} \int_{V^{(e)}} \left[N^{(e)}\right]^T (2\mu\theta) dV \, , \tag{4}$$

where $\left\{g^{(e)}\right\} = \begin{bmatrix} \dfrac{\partial N_i^{(e)}}{\partial x} & \dfrac{\partial N_j^{(e)}}{\partial x} & \cdots & \dfrac{\partial N_r^{(e)}}{\partial x} \\ \dfrac{\partial N_i^{(e)}}{\partial y} & \dfrac{\partial N_j^{(e)}}{\partial y} & \cdots & \dfrac{\partial N_r^{(e)}}{\partial y} \end{bmatrix} \begin{Bmatrix} \Phi_i \\ \Phi_j \\ \cdots \\ \Phi_r \end{Bmatrix}$,

$\lfloor N_i, N_j, ..., N_r \rfloor$ - FE shape functions,

$\{\Phi_i, \Phi_j, ..., \Phi_r\}$ - nodal values of the stress function.

The torque is proportional to the volume encompassed by the stress surface:

$$M = 2\iint_\Omega \varphi \, dxdy \, , \tag{5}$$

where Ω – is the cross-section of the rod.

Due to the symmetry of the region, the solution in (Ilyin, Karpov and Maslennikov, 1990) is performed on 1/8 part of the square shaded in Fig. 1. The partitioning of the shaded area is performed by triangulation (Fig. 2).

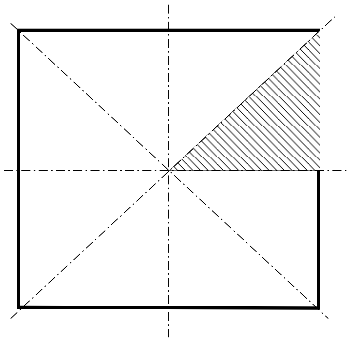

Fig 1. Cross-section of the rod square

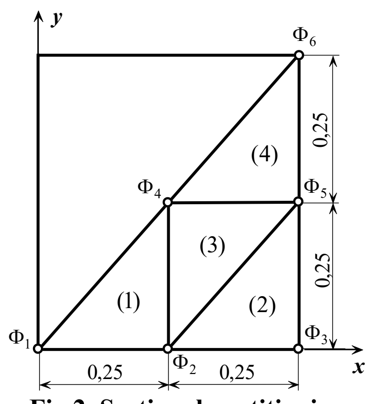

Fig 2. Sectional partitioning to triangular FE

As a result, an approximate solution is obtained, shown in Fig. 3 (Segerlind, 1979).

In (Timoshenko, 1972), this problem was solved by the Ritz method. Fig. 4 shows the stress surface, which corresponds to the solution for a square with cross-sectional area $1\,cm^2$, if the function in the form of an infinite trigonometric series is taken as an approximating function (Timoshenko, 1972):

$$\varphi = \frac{32\mu\theta b^2}{\pi^4} \sum_{m=1,3,\ldots} \sum_{n=1,3,\ldots} \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{m\,n\left(m^2\alpha^2 + n^2\right)}; \tag{6}$$

$$M = \frac{32\mu\theta}{\pi^4} \frac{8ab^3}{\pi^2} \sum_{m=1,3,\ldots} \sum_{n=1,3,\ldots} \frac{1}{m^2 n^2\left(m^2\alpha^2 + n^2\right)}; \tag{7}$$

In the following we will call the solution (6)-(7) exact.

Even a visual comparison of Fig. 3 and Fig. 4 shows the poor accuracy of the solution obtained using 4 triangles. The disadvantage of using linear interpolation polynomials is the inability to obtain gradients as functions of coordinates. To refine the values of the stresses inside the rod, we can do the following: 1) increase the number of triangular elements; 2) take triangles with a larger number of nodes on the boundary; 3) apply finite elements in the form of a square.

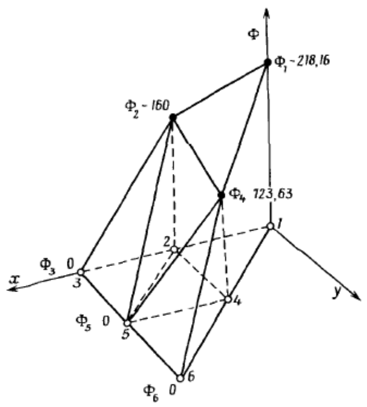

Fig 3. Nodal values of the stress function at triangulation

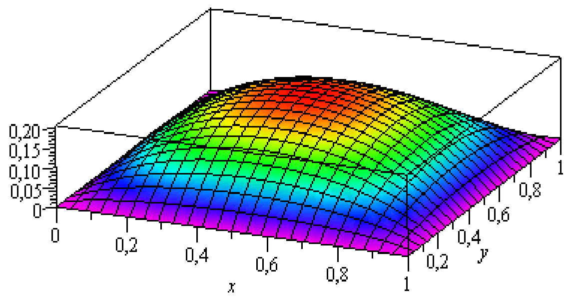

Fig 4. Stress surface, given by formula (7) (vertical coordinate scale 1:1000)

To refine the stress values, the authors of the paper applied the procedure of the biquadratic finite element method (SFE Q8) (Fig. 5). The partitioning of $1/4$ of the rod section into squares is shown in Fig. 6.

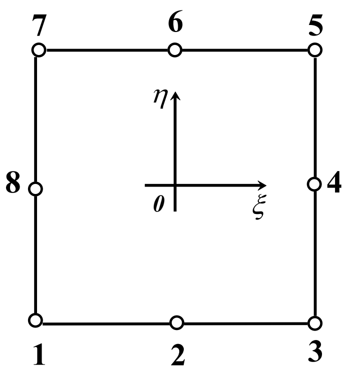

Fig 5. SFE Q8 ($|\xi| \le 1, |\eta| \le 1$)

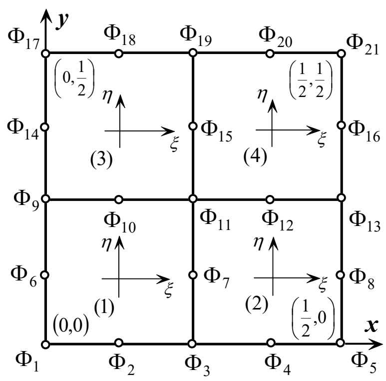

Fig 6. Sectional partitioning to biquadratic FE

Alternative basis functions with the control parameter p (Astionenko, Litvinenko and Khomchenko, 2009) are obtained for the biquadratic FE by the combined algebro-geometric method:

$$N_i = \frac{1}{16}(1+\xi_i\xi)(1+\eta_i\eta)[(36p-1)(1-\xi_i\xi-\eta_i\eta)+(36p+3)\xi_i\xi\eta_i\eta], \tag{8}$$

$$i = 1,3,5,7\ ; \qquad \xi_i, \eta_i = \pm 1\ .$$

$$N_i = \frac{1}{16}(1-\xi^2)(1+\eta_i\eta)[(5-36p)+(36p+3)\eta_i\eta], \tag{9}$$

$$i = 2,6\ ; \qquad \eta_i = \pm 1\ .$$

$$N_i = \frac{1}{16}(1-\eta^2)(1+\xi_i\xi)[(5-36p)+(36p+3)\xi_i\xi], \tag{10}$$

$$i = 4, 8\ ;\ .\ \xi_i = \pm 1$$

The existence of the exact solution allows us to estimate the relative error, which is obtained by the FEM calculation with biquadratic elements. The relative error for the stress function is calculated by the formula:

$$\Delta\varphi = \frac{1}{n}\sum_i \frac{|\varphi_i - \Phi_i|}{\varphi_i}\cdot 100\%\ , \tag{11}$$

where φ_i is the exact value obtained using formula (6);

Φ_i - approximate value ($i = 1;2;6;7;9;10;11;12;14;15$).

The relative error in the torque calculation was calculated using a similar formula:

$$\Delta M = \frac{|M - M_p|}{M}\cdot 100\%\ , \tag{12}$$

where M is the exact value obtained using formula (7);

M_p - approximate value.

The errors for the stress and torque functions that are obtained by FEM using alternative functions at different values of parameter p are summarized in Table 1.

Table 1

Value of parameter *p*	***Significance*** M	**Relative error,** ΔM **, %**	**Relative error,** $\Delta\varphi$ **, %**
-0,50	193,928	1,166	3,950
-0,375	194,417	0,916	3,449
-0,25	195,113	0,562	2,559
-0,20	195,448	0,391	1,989
-0,125	195,881	0,170	0,704
-0,105	**195,917**	**0,152**	0,461
-0,10	195,914	0,153	0,439
Standard -1/12=-0,08(3)	195,855	0,183	0,409
-0,075	195,789	0,217	0,586
-0,05	195,362	0,435	1,371
-0,025	194,377	0,937	2,168
0	192,423	1,933	2,722
0,025	189,091	3,631	2,433
0,04	186,540	4,931	1,521

0,05	184,819	5,808	0,568
1/18=0,0(5)	183,924	6,264	**0,110**
0,065	182,591	6,943	1,219
0,075	181,519	7,490	2,485
0,10	180,415	8,052	5,177
0,125	180,785	7,864	6,808
0,25	184,974	5,729	8,062
0,375	187,210	4,589	7,571
0,50	188,393	3,986	7,180

Fig. 7 shows the dependence of the relative error in the calculation of torque M and stress function φ on the parameter *p*. The graph of the stress function (round points in Fig. 7) shows that when changing the parameter $p \in [-0{,}5;0{,}5]$, the minimum is reached twice: at $p = -1/12$, which corresponds to the standard SCE-8 basis (Segerlind, 1979), and at $p = 1/18$ (this basis corresponds to the stiffness matrix with a minimum trace (Sekulovich, 1993). The solution obtained at $p = 1/18$, is four times more accurate.

For the relative error in the torque calculation (curve shown by rhombuses in Fig. 7), when the parameter $p \in [-0{,}5;0{,}5]$ is changed, the minimum is reached once: at $p = -21/200 = -0{,}105$. The standard basis gives a solution of the same order of accuracy.

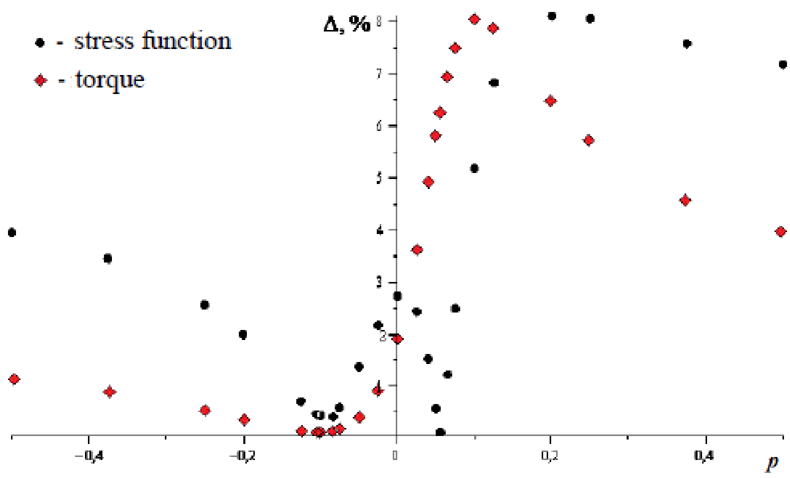

Fig 7. Dependences of the calculation error of the stress function and torque from the parameter p

The monotonic increase of the relative torque error in the interval $(-0{,}1;0{,}1)$ is puzzling, because the authors hoped to find a second minimum in this interval. The reason for this behavior of the error has not yet been established.

Conclusions

It has been proved that in the problem of torsion of a square section rod by the finite element method, new alternative models of the biquadratic finite element allow obtaining higher accuracy in comparison with the known standard basis of this element. The alternative SCE-8 bases also have an advantage over the traditional triangulation procedure in FEM, because more triangular finite elements must be used to obtain the specified accuracy. In addition, unlike triangular FEs, alternative serendipitous models allow us to obtain tangential stresses as functions of coordinates.

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